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Some Model Theory of Free Groups and Free Algebras

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13. ABSTRACT (*Maximum 200 words*)

The subject of this paper is group theory and logic with an emphasis on groups. The concept of a group is one of the most powerful and unifying of all concepts in mathematics. Moveover in addition to turning up in every branch of mathematics, groups have endless applications in science. Wherever there is symmetry, there is a group. The Lorentz transformations of relativity from a Lie group based on continuous rotation of an object in space-time. Finite groups underlie the structure of all crystals and are indispensable in chemistry, quantum mechanics, and particle physics. The famous eight fold way, which classifies the family of subatomic particles known as hadrons, is a Lie group.

In view of the great elegance and utility of groups, it is understandable that mathematicians study them and their properties, which is precisely what this paper does. There is, of course, no way to predict what other practical application this material will have. We do, however, know that groups lie at the very heart of the structure of the universe.

14. SUBJECT TERMS

group	free algebra	positive sentence	strong discrimination
free group	universal formula	negative sentence	non-Abelian free group
algebra	existential formula	(B+3N)- discrimination	model class free product with amalgamation

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CONTENTS

SUMMARY

This paper is an expansion of earlier work in which (using the notation of Bell and Slomsom, *Models and Ultraproducts*) A $\forall$ B and A $\subseteq_{\forall}$ B were characterized, where A and B are free groups in some non-trivial variety. Here we extract sufficient conditions for A $\forall$ B and A $\subseteq_{\forall}$ B for arbitrary algebras. We apply these to show that if F is a non-Abelian free group and $K = (F * \bar{F} \ ; \ u = \bar{u})$ is the "Baumslag construct," $F \cong \bar{F}$ under the map $f \longrightarrow \bar{f}$ and u is not a proper power in F, then

$$F \subseteq_{\forall} K.$$

SOME MODEL THEORY OF FREE GROUPS AND FREE ALGEBRAS

1. Introduction

This paper treats some aspects of the model theory of groups and algebras free in a variety. Specifically we study the persistence of universal and existential formulae and make some observations about positive and negative sentences as well. The paper falls naturally into three parts.

In the first part, the concepts of (B+3N)-discrimination and strong discrimination are elucidated. Complete proofs are given for some of the results announced in [10] as well as an outline of the proof of Theorem 4 also announced in [10]. In the second part, we give ourselves a non-Abelian free group F and consider its relation to the Baumslag construct $K = (F * \bar{F};\ u = \bar{u})$ where u is not a proper power in F. We ponder the model class of the theory of the non-Abelian free groups in this part. In the third part, we consider the equation $[y_1,y_2] = x_1^2x_2^2$ in a free group and its implications for questions arising from the the second part. We conclude with a list of questions which to the best of our knowledge remain open.

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2 .Definitions and Notations

Let ω be the set of nonnegative integers and $\mathbb{N}$ be the set of positive integers. Let $\mathbb{Z}$ be the set of all integers. An *operator domain* is an ordered pair $(\mathcal{O},a)$ where $\mathcal{O}$ is a set and $a : \mathcal{O} \longrightarrow \omega$ is a function. If Ω is an operator domain, let L_Ω be the first-order language with equality containing each $c \in \mathcal{O}$ such that $a(c) = 0$ as a constant symbol and each $f \in \mathcal{O}$ such that $a(f) = n \in \mathbb{N}$ as a function symbol of degree n. Structures appropriate for L_Ω are *Ω-algebras*. A formula of the type $s = t$ where s and t are terms of L_Ω is a *law* for Ω-algebras.

An Ω-algebra *satisfies* the law $s = t$ just in case equality holds whenever arbitrary elements of the algebra are substituted for the free variables. A class $\mathcal{V}$ of Ω-algebras is a *variety* of Ω-algebras just in case there is at least one set Λ of laws such that $\mathcal{V}$ is precisely the model class of Λ. By a classic theorem of Garrett Birkhoff, a class $\mathcal{V}$ of Ω-algebras is a variety if and only if it is closed under the formation of subalgebras, homomorphic images, and unrestricted direct products. We shall say that a variety $\mathcal{V}$ of Ω-algebras is *nontrivial* if it contains at least one algebra with at least two elements. Nontrivial varieties contain free algebras of every rank $r \geq 1$. A cardinal is an ordinal not equipotent with any prior ordinal. If $r \geq 1$ is a cardinal and $\mathcal{V}$ is a nontrivial variety of Ω-algebras, then $F_r(\mathcal{V})$ is an Ω-algebra free of rank r in $\mathcal{V}$.

Let $P \subseteq \mathbb{N}$ be the set of primes. For each subset $P_o \subseteq P$ let $\Omega(P_o)$ be the operator domain consisting of a binary operator $\cdot$, a unary operator $^{-1}$, a constant symbol 1, and for each $p \in P_o$ a

unary operator $^{1/p}$. A $D(P_o)$-algebra shall be an $\Omega(P_o)$-algebra satisfying the laws

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$x \cdot 1 = x$$

$$x \cdot x^{-1} = 1$$

$$\left(x^{1/p} \right)^p = x \text{ for all } p \in P_o$$

$$\left(x^p \right)^{1/p} = x \text{ for all } p \in P_o$$

In particular, if $P_o = \phi$, then a $D(P_o)$-algebra is just a group. In the language of G. Baumslag [3], a $D(P_o)$-algebra is a D_{P_o}-group. $L_0 = L_{\Omega(\phi)}$ shall be, "The language of group theory." We shall find it convenient to commit the abuses of adjoining constants from an algebra (rather than a disjoint set of formal symbols) to our formal languages and of not distinguishing notationally between an algebra and its underlying set. Let an operator domain Ω be given. Let $n \in \mathbb{N}$. A *Π_n-sentence* of L_Ω is one logically equivalent to one of the form $\forall\bar{x}\exists\bar{y}\forall\bar{z}\ldots\psi(\bar{x},\bar{y},\bar{z},\ldots)$ where $\bar{x}$, $\bar{y}$, $\bar{z}$, ... are tuples of variables, ψ is a formula of L_Ω containing no quantifiers, the leftmost block of quantifiers are universal, and there are at most n-1 alternations of quantifier type. A *Σ_n-sentence* of L_Ω is one logically equivalent to one of the form $\exists\bar{x}\forall\bar{y}\exists\bar{z}\ldots\psi(\bar{x},\bar{y},\bar{z},\ldots)$ where $\bar{x}$, $\bar{y}$, $\bar{z}$,... are tuples of variables, ψ is a formula of L_Ω containing no quantifiers, the leftmost block of quantifiers are existential, and there are at most n-1 alternations of quantifier type. A Π_1-sentence is a *universal sentence* and a Σ_1-sentence is an *existential sentence.* A *positive sentence* is one logically equivalent to one constructed recursively from $\wedge$, $\vee$, $\forall$, and $\exists$, but containing no occurrences of

$\sim$, $\Rightarrow$, or $\Leftrightarrow$. A *negative sentence* is one which is logically equivalent to the negation of a positive sentence. If A and B are Ω-algebras we shall write $A \equiv B$ to indicate that A and B are elementarily equivalent (i.e., satisfy the same sentences of L_Ω). If A is a subalgebra of B and the inclusion map embeds A elementarily in B, we shall write $A \propto B$ to so indicate that state of affairs. (See [7] or [11] for the definition of an elementary embedding.) Following Bell and Slomson, we shall write given an algebra B and a subalgebra A of B the expressions $A\forall B$ and $A \leq_\forall B$ to indicate that A and B satisfy precisely the same universal sentences and that A and B satisfy precisely the same universal formulas of L_Ω (i.e., universal sentences in the language obtained from L_Ω by adjoining the elements of A as constants) respectively.

If G is a group and X is a subset of G, then gp(X) shall be the subgroup of G generated by X. If $(x,y) \in G^2$, then their *commutator* $[x,y]$ is the element $x^{-1}y^{-1}xy$. The *simple commutator* $[x_1,\dots,x_n]$ is defined inductively by

$$\begin{cases} x_1 = x_1 \\ [x_1,\dots,x_n,x_{n+1}] = [[x_1,\dots,x_n],x_{n+1}] \end{cases}$$

If $c \in \mathbb{N}$, then the variety of all nil-c groups $\mathcal{N}_c$ is the class of all groups satisfying the law $[x_1,\dots,x_{c+1}] = 1$. Equivalently if the *lower central series* of G is defined inductively by

$$\begin{cases} \gamma_1(G) = G \\ \gamma_{n+1}(G) = gp(\ [x,y] \mid x \in \gamma_n(G),\ y \in G) \end{cases}$$

then $\mathcal{N}_c$ is precisely the class of all groups G for which $\gamma_{c+1}(G) = 1$. We let $\mathcal{A} = \mathcal{N}_1$ be the variety of all Abelian groups.

If $\mathcal{U}$ and $\mathcal{V}$ are varieties of groups, then the class $\mathcal{UV}$ of all extensions of elements of $\mathcal{U}$ by elements of $\mathcal{V}$ is a variety called

the product variety of $\mathcal{U}$ by $\mathcal{V}$. Thus $\mathcal{A}^2$ is the variety of all metabelian groups. Furthermore, $\mathcal{N}_c \wedge \mathcal{A}^2$ is the variety of all groups simultaneously metabelian and nil-c. If $n \in \mathbb{N}$, then $\mathcal{B}_n$ is the *Burnside variety of expcnent n* determined by the law $x^n = 1$. $\mathcal{O}$ shall be the variety of all groups and $\mathcal{E}$ the *trivial variety* of groups (i.e., the isomorphism class of the one element group.)

3. Are Some Groups More Discriminating than Others?

Let r and s be cardinals with $r \leq s$. Let $\{a_\alpha \mid \alpha < s\}$ be a set of s distinct symbols. Suppose $\mathcal{V}$ is a nontrivial variety of Ω-algebras. Let $F_s(\mathcal{V})$ be freely generated in $\mathcal{V}$ by $\{a_\alpha \mid \alpha < s\}$ and suppose $F_r(\mathcal{V})$ is the subalgebra generated (necessarily freely) by the initial segment $\{a_\alpha \mid \alpha < r\}$. Then Vaught has shown that $F_r(\mathcal{V}) \propto F_s(\mathcal{V})$ whenever r (and therefore also s) is infinite. See Grätzer [11], Theorem 4, p. 237. In particular, $F_r(\mathcal{V}) \equiv F_s(\mathcal{V})$ whenever r (and therefore also s) is infinite. However, in general, free algebras of finite rank can be distinguished by first-order sentences. For example,

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 \left[\bigwedge_{i \not\equiv j} \sim\exists y(x_i x_j^{-1} = y^2) \wedge \forall z \exists w \left(\bigvee_{i=1}^{4} z x_i^{-1} = w^2 \right) \right]$$

picks out precisely $F_2(\mathcal{A}) \cong \mathbb{Z}^2$ in the variety $\mathcal{A}$ of all Abelian groups. However, Tarski has conjectured that $F_r(\mathcal{O}) \propto F_s(\mathcal{O})$ (in particular $F_r(\mathcal{O}) \equiv F_s(\mathcal{O})$) whenever $2 \leq r \leq s$.

Before proceeding with our own definitions and results, we pause to assert from the literature

Theorem (Lyndon [16]): Positive sentences of L_Ω are preserved under homomorphic images.

Theorem (Sacerdote [22]): If $2 \leq r \leq s \leq \omega$, then $F_r(\mathcal{O})$ and

$F_s(O)$ are such that every Π_3-sentence in the language of $F_r(O)$ which holds in $F_s(O)$ must also hold in $F_r(O)$. In particular, $F_r(O)$ $\le_V F_s(O)$.

Corollary (Sacerdote [22]): If $2 \le r \le s \le \omega$, then $F_r(O)$ and $F_s(O)$ satisfy precisely the same Π_2 and Σ_2-sentences in the language of $F_r(O)$.

Theorem (Sacerdote [23]): If G and H are nontrivial free products neither isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, then G and H satisfy precisely the same positive sentences of L_o. In particular, the non-Abelian absolutely free groups satisfy precisely the same positive and negative sentences of L_o - a result due originally to Merzljakov [18]).

Definition 1: Let $\mathcal{V}$ be a nontrivial variety of Ω-algebras and $r \ge 1$ a cardinal. $F_r(\mathcal{V})$ *(B+3N)-discriminates* $\mathcal{V}$ provided, whenever (s_i, t_i) are finitely many pairs of terms in the language L_Ω such that none of the equations $s_i = t_i$ is a law of $\mathcal{V}$, there are elements $b_0, b_1, \ldots \in F_r(\mathcal{V})$ such that simultaneously $s_i(b_0, b_1, \ldots)$ $\ne t_i(b_0, b_1, \ldots)$ for all i.

We remark that B+3N refers to the authors of [6] in which this concept is introduced for groups.

Definition 2: Let $\mathcal{V}$ be a nontrivial variety of Ω-algebras and $r \in \mathbb{N}$. $F_r(\mathcal{V})$ *strongly discriminates* $\mathcal{V}$ just in case it possesses an ordered set $a_0 < a_1 < \ldots < a_{r-1}$ of $\mathcal{V}$-free generators such that whenever (s_i, t_i) are finitely many pairs of terms of L_Ω with none of the equations $s_i = t_i$ a law in $\mathcal{V}$, then there are elements $b_r, b_{r+1}, \ldots$ $\in F_r(\mathcal{V})$ such that simultaneously $s_i(a_0, \ldots, a_{r-1}, b_r, b_{r+1}, \ldots) \ne$ $t_i(a_0, \ldots, a_{r-1}, b_r, b_{r+1}, \ldots)$ for all i.

Definition 3: Let A and B be Ω-algebras with A a subalgebra

of B. The ordered pair (A,B) satisfies the *strong residual property* provided for every finite set $u_i \neq v_i$ of unequal elements of B there exists at least one retraction $h : B \longrightarrow A$ such that simultaneously $h(u_i) \neq h(v_i)$ for all i.

Evidently $F_r(\mathcal{V})$ strongly discriminates $\mathcal{V}$ implies $F_r(\mathcal{V})$ (B+3N)-discriminates $\mathcal{V}$ and $F_r(\mathcal{V})$ (B+3N)-discriminates $\mathcal{V}$ implies $F_r(\mathcal{V})$ generates $\mathcal{V}$. There are examples of groups generating a variety but not discriminating it.(E.g., no finite group $G \neq 1$ can be discriminating.)

Lemma 1: Let $\mathcal{V}$ be a nontrivial variety of Ω-algebras and $r \in \mathbb{N}$. Then the following three statements are pairwise equivalent:

(1) $F_r(\mathcal{V})$ strongly discriminates $\mathcal{V}$.

(2) For every finite s with $s \geq r$, the pair $(F_r(\mathcal{V}),F_s(\mathcal{V}))$ satisfies the strong residual property.

(3) For every s with $s \geq r$, the pair $(F_r(\mathcal{V}),F_s(\mathcal{V}))$ satisfies the strong residual property.

Proof: (1) $\Rightarrow$ (2). Let (u_i, v_i) be finitely many pairs of unequal elements of $F_s(\mathcal{V})$. Express

$$u_i = s_i(a_0,\dots,a_{s-1}) \text{ and } v_i = t_i(a_0,\dots,a_{s-1})$$

as terms on the free generators of $F_s(\mathcal{V})$. Then none of the equations

$$s_i(x_0,\dots,x_{s-1}) = t_i(x_0,\dots,x_{s-1})$$

is a law in $\mathcal{V}$. Thus there are elements $b_r,\dots,b_{s-1} \in F_r(\mathcal{V})$ such that simultaneously

$$s_i(a_0,\dots,a_{r-1},b_r,\dots,b_{s-1}) \neq t_i(a_0,\dots,a_{r-1},b_r,\dots,b_{s-1})$$

in $F_r(\mathcal{V})$. The map on the generators

$$\begin{cases} a_n \longmapsto a_n & \text{if } 0 \leq n < r \\ a_n \longmapsto b_n & \text{if } r \leq n < s \end{cases}$$

extends to a retraction $\psi : F_s(\mathcal{V}) \longrightarrow F_r(\mathcal{V})$ such that

$$\psi(u_i) \neq \psi(v_i) \text{ for all } i.$$

(2) $\Rightarrow$ (3). Suppose now that s is infinite and $A = \{a_\alpha \mid \alpha < s\}$ is a set of free generators for $F_s(\mathcal{V})$. Let $A_o = \{a_n \mid n < \omega\}$ and $A_1 = \{a_\alpha \mid \omega \leq \alpha < s\}$. Suppose (u_i,v_i) are finitely many pairs of unequal elements of $F_s(\mathcal{V})$. Express $u_i = s_i(a_\alpha)$, $v_i = t_i(a_\alpha)$ as terms on the generators. Let A_{oo} be the set of all $a_n \in A_o$ involved in at least one of $s_i(a_\alpha)$ or at least one $t_i(a_\alpha)$, and A_{11} be the set of all $a_\alpha \in A_1$ involved in at least one $s_i(a_\alpha)$ or at least one $t_i(a_\alpha)$. Then since A_{11} is finite and $A_o - A_{oo}$ is infinite, there is an embedding $f : A_{11} \longrightarrow A_o - A_{oo}$. Let A_{o1} be the image of A_{11}. Now $A_{oo} \cup A_{o1}$ is finite so that there is some finite $N \geq r-1$ such that $A_{oo} \cup A_{o1} \subseteq \{a_0,\ldots,a_N\}$.

Now the map

$$\begin{cases} a_n \longmapsto a_n & \text{if } a_n \in \{a_0,\ldots,a_N\} \\ a_n \longmapsto a_0 & \text{if } a_n \in A_o - \{a_0,\ldots,a_N\} \\ a_\alpha \longmapsto f(a_\alpha) & \text{if } a_\alpha \in A_{11} \\ a_\alpha \longmapsto a_0 & \text{if } a_\alpha \in A_1 - A_{11} \end{cases}$$

extends to a retraction $\Psi : F_s(\mathcal{V}) \longrightarrow F_{N+1}(\mathcal{V})$. Moreover, all the u_i, v_i lie in the subalgebra generated by $(\{a_0,\ldots,a_N\}-A_{o1}) \cup A_{11}$ which is mapped isomorphically by the restriction of Ψ onto the subalgebra generated by $\{a_0,\ldots,a_N\}$ - namely $F_{N+1}(\mathcal{V})$. Thus, $\Psi(u_i) \neq \Psi(v_i)$ foe all i. Now since $N+1 \geq r$ is finite there is a retraction$\phi : F_{N+1}(\mathcal{V}) \longrightarrow F_r(\mathcal{V})$ such that

$$\phi(\Psi(u_i)) \neq \phi(\Psi(v_i)) \text{ for all } i.$$

Thus $\phi\Psi$ is the required retraction.

(3) $\Rightarrow$ (1). Let $s_i(x_0,\ldots,x_{s-1}) \neq t_i(x_0,\ldots,x_{s-1})$ be finitely many "non-laws" of $\mathcal{V}$. We may assume $s \geq r$. Then

$$s_i(a_0,\ldots,a_{s-1}) \neq t_i(a_0,\ldots,a_{s-1}) \text{ in } F_s(\mathcal{V})$$

where $F_s(\mathcal{V})$ is freely generated by $a_0,\ldots,a_{s-1}$. Then there is a retraction $\phi : F_s(\mathcal{V}) \longrightarrow F_r(\mathcal{V})$ such that simultaneously

$s_i(a_0,\ldots,a_{r-1},\phi(a_r),\ldots,\phi(a_{s-1})) \neq t_i(a_0,\ldots,a_{r-1},\phi(a_r),\ldots,\phi(a_{s-1}))$

in $F_r(\mathcal{V})$. Thus $F_r(\mathcal{V})$ strongly discriminates $\mathcal{V}$. ■

Lemma 2: Let A and B be Ω-algebras with A a subalgebra of B. If the ordered pair (A,B) satisfies the strong residual property, then $A \leq_{\forall} B$.

Proof: Much as the arguments involved in Robinson's Test ([7] and [20]), it suffices to show that every *primitive sentence* in the language of A which holds in B must also hold in A. Such a sentence is one of the form

$$\exists\, \bar{x} \left[\bigwedge_i (p_{i1}(\bar{a},\bar{x}) = p_{i2}(\bar{a},\bar{x})) \wedge \bigwedge_j (q_{j1}(\bar{a},\bar{x}) \neq q_{j2}(\bar{a},\bar{x}))\right]$$

where $\bar{a} = (a_1,a_2,\ldots)$ is a tuple of elements of A, $\bar{x} = (x_1,x_2,\ldots)$ is a *tuple of variables and* $p_{i\nu}$, $q_{i\nu}$ *(ν = 1,2) are terms of* L_Ω. Suppose this primitive sentence is true in B. Then there is a tuple $\bar{b} = (b_1,b_2,\ldots)$ of elements of B such that

$$\bigwedge_i (p_{i1}(\bar{a},\bar{b}) = p_{i2}(\bar{a},\bar{b})) \wedge \bigwedge_j (q_{j1}(\bar{a},\bar{b}) \neq q_{j2}(\bar{a},\bar{b}))$$

is true in B. There exists a retraction $h\colon B \longrightarrow A$ such that

$$h(q_{j1}(\bar{a},\bar{b})) \neq h(q_{j2}(\bar{a},\bar{b}))$$

for all j. Applying h, we see that

$$\bigwedge_i (p_{i1}(\bar{a},h(\bar{b})) = p_{i2}(\bar{a},h(\bar{b}))) \wedge \bigwedge_j (q_{j1}(\bar{a},h(\bar{b})) \neq q_{j2}(\bar{a},h(\bar{b})))$$

is true in A. Therefore,

$$\exists\, \bar{x} \left[\bigwedge_i (p_{i1}(\bar{a},\bar{x}) = p_{i2}(\bar{a},\bar{x})) \wedge \bigwedge_j (q_{j1}(\bar{a},\bar{x}) \neq q_{j2}(\bar{a},\bar{x}))\right]$$

is true in A.■

In fact, we have

Theorem 1: If $\mathcal{V}$ is a nontrivial variety of Ω-algebras and $r \geq 1$ is a cardinal, then $F_r(\mathcal{V}) \equiv_{\forall} F_s(\mathcal{V})$ for all s with $s \geq r$ if and only if $F_r(\mathcal{V})$ (B+3N)-discriminates $\mathcal{V}$.

and

Theorem 2: If $\mathcal{V}$ is a nontrivial variety of Ω-algebras and $r \in \mathbb{N}$, then $F_r(\mathcal{V}) \leq_{\mathcal{V}} F_s(\mathcal{V})$ for all s with $s \geq r$ if and only if $F_r(\mathcal{V})$ strongly discriminates $\mathcal{V}$.

In particular, if $F_r(\mathcal{V})$ discriminates $\mathcal{V}$, then $F_r(\mathcal{V})$ not only satisfies no more laws than the $F_s(\mathcal{V})$ for $s \geq r$, but no more universal sentences of any kind.

Proof of Theorem 1: Suppose first that $F_r(\mathcal{V}) \forall F_s(\mathcal{V})$ for all s with $s \geq r$ to show that $F_r(\mathcal{V})$ (B+3N)-discriminates $\mathcal{V}$. Let (s_i,t_i) be finitely many pairs of terms of L_Ω such that none of the equations $s_i = t_i$ is a law in $\mathcal{V}$. Then the terms s_i and t_i involve only finitely many free variables - say $x_0,\ldots,x_{N-1}$ where we may assume that $N \geq r$. Then $s_i(a_0,\ldots,a_{N-1}) \neq t_i(a_0,\ldots,a_{N-1})$ in $F_N(\mathcal{V})$ where $\{a_0,\ldots,a_{N-1}\}$ freely generates $F_N(\mathcal{V})$. Thus

$$\exists x_0 \ldots \exists x_{N-1} \left[\bigwedge_i (s_i(x_0,\ldots,x_{N-1}) \neq t_i(x_0,\ldots,x_{N-1})) \right]$$

is true in $F_N(\mathcal{V})$.

Since $F_r(\mathcal{V})$ and $F_N(\mathcal{V})$ satisfy the same universal sentences, they must satisfy the same existential sentences as well. Therefore, $\exists x_0 \ldots \exists x_{N-1} \left[\bigwedge_i (s_i(x_0,\ldots,x_{N-1}) \neq t_i(x_0,\ldots,x_{N-1})) \right]$ holds also in $F_r(\mathcal{V})$. Since (s_i,t_i) was arbitrary, we conclude that $F_r(\mathcal{V})$ (B+3N)-discriminates $\mathcal{V}$.

Now suppose that $F_r(\mathcal{V})$ (B+3N)-discriminates $\mathcal{V}$ in order to show that $F_r(\mathcal{V}) \forall F_s(\mathcal{V})$ for all s with $s \geq r$. If r is infinite, then because of Vaught's theorem, there is nothing to prove. Suppose first that both r and s are finite. Consider the primitive sentence

$$\exists \bar{x} \left[\bigwedge_i (p_{i1}(\bar{x}) = p_{i2}(\bar{x})) \wedge \bigwedge_j (q_{j1}(\bar{x}) \neq q_{j2}(\bar{x})) \right] \qquad (*)$$

of L_Ω. Suppose that (*) holds in $F_s(\mathcal{V})$. Then there is a tuple $\bar{c}$ =

$(c_1,\ldots,c_k)$ of elements of $F_s(V)$ such that

$$\bigwedge_i (p_{i1}(\bar{c}) = p_{i2}(\bar{c})) \wedge \bigwedge_j (q_{j1}(\bar{c}) \neq q_{j2}(\bar{c})) \qquad (**)$$

is true in $F_s(V)$.

Expressing $c_\nu = u_\nu(a_0,\ldots,a_{s-1})$, $1 \le \nu \le k$, as a term on the generators, we have that simultaneously

$$p_{i1}(u_1(x_0,\ldots,x_{s-1}),\ldots,u_k(x_0,\ldots,x_{s-1})) =$$
$$p_{i2}(u_1(x_0,\ldots,x_{s-1}),\ldots,u_k(x_0,\ldots,x_{s-1}))$$

are laws of V and

$$q_{i1}(u_1(x_0,\ldots,x_{s-1}),\ldots,u_k(x_0,\ldots,x_{s-1})) \neq$$
$$q_{i2}(u_1(x_0,\ldots,x_{s-1}),\ldots,u_k(x_0,\ldots,x_{s-1}))$$

are "non-laws" of V. Thus, there are elements $b_0,\ldots,b_{s-1} \in F_r(V)$ such that simultaneously

$$q_{j1}(u_1(b_0,\ldots,b_{s-1}),\ldots,u_k(b_0,\ldots,b_{s-1})) \neq$$
$$q_{j2}(u_1(b_0,\ldots,b_{s-1}),\ldots,u_k(b_0,\ldots,b_{s-1}))$$

in $F_r(V)$ for all j. Moreover,

$$p_{i1}(u_1(b_0,\ldots,b_{s-1}),\ldots,u_k(b_0,\ldots,b_{s-1})) =$$
$$p_{i2}(u_1(b_0,\ldots,b_{s-1}),\ldots,u_k(b_0,\ldots,b_{s-1}))$$

in $F_r(V)$ for all i, since

$$p_{i1}(\bar{u}(x_0,\ldots,x_{s-1})) = p_{i2}(\bar{u}(x_0,\ldots,x_{s-1}))$$

is a law in V for all i. Hence

$$\bigwedge_i \left[p_{i1}(\bar{u}(b_0,\ldots,b_{s-1})) = p_{i2}(\bar{u}(b_0,\ldots,b_{s-1}))\right] \wedge$$
$$\bigwedge_j \left[q_{j1}(\bar{u}(b_0,\ldots,b_{s-1})) \neq q_{j2}(\bar{u}(b_0,\ldots,b_{s-1}))\right] \qquad (***)$$

is true in $F_r(V)$. It follows

$$\exists\, \bar{x} \left[\bigwedge_i (p_{i1}(\bar{x}) = p_{i2}(\bar{x})) \wedge \bigwedge_j (q_{j1}(\bar{x}) \neq q_{j2}(\bar{x}))\right] \qquad (*)$$

is true in $F_r(V)$. Hence, $F_r(V) \not\equiv F_s(V)$.

The one remaining case is r finite and s infinite. Suppose

$$\exists\, \bar{x} \left[\bigwedge_i (p_{i1}(\bar{x}) = p_{i2}(\bar{x})) \wedge \bigwedge_j (q_{j1}(\bar{x}) \neq q_{j2}(\bar{x}))\right] \qquad (*)$$

holds in $F_s(V)$. Then the elements asserted to exist by (*) can be

expressed as terms on finitely many free generators, say $a_{\alpha_1},\ldots,a_{\alpha_N}$. But then (*) holds in the subalgebra of $F_s(\mathcal{V})$ generated by $a_{\alpha_1},\ldots,a_{\alpha_N}$. We may assume that $N \geq r$. This subalgebra is isomorphic to $F_N(\mathcal{V})$, and so we have reduced this case to the previous one. ■

The proof of Theorem 2 is similar.

We first observe that (B+3N)-discrimination and strong discrimination are inherited by free algebras of higher rank (the former is obvious and we shall presently see why the latter is so). We next note that strong discrimination implies (B+3N)-discrimination. Thus in order to show for a particular variety $\mathcal{V}$ (B+3N)-discriminated by one of its free algebras of finite rank that these two concepts coincide for $\mathcal{V}$, it suffices to show that if

$$m = \min \{ n \in \mathbb{N} \mid F_n(\mathcal{V}) \text{ (B+3N)-discriminates } \mathcal{V} \},$$

then

$$F_m(\mathcal{V}) \text{ strongly discriminates } \mathcal{V}.$$

That strong discrimination is inherited by free algebras of higher (finite) rank is a simple consequence of the following characterization of strong discrimination: Let $\mathcal{V}$ be a nontrivial variety of Ω-algebras and let $r \in \mathbb{N}$. Then $\mathcal{V}$ is strongly discriminated by its free algebra of rank r provided that the following holds. For every finite set of ordered pairs $(s_i(x_0,\ldots,x_n),t_i(x_0,\ldots,x_n))$, where $n \geq r$, of terms of L_Ω such that none of the equations $s_i = t_i$ is a law in $\mathcal{V}$, there are terms $u_\nu = u_\nu(x_0,\ldots,x_{r-1})$ for $r \leq \nu \leq n$ of L_Ω such that simultaneously none of the equations

$$s_i(x_0,\ldots,x_{r-1},u_r,\ldots,u_n) = t_i(x_0,\ldots,x_{r-1},u_r,\ldots,u_n)$$

is a law in $\mathcal{V}$.

Lemma 3: $\mathbb{Z} = F_1(\mathcal{A})$ **strongly discriminates** $\mathcal{A}$.

Proof: This is only a slight modification of the proof in H. Neumann (see p.29 of [19]) that $\mathbb{Z}$ (B+3N)-discriminates $\mathcal{A}$. For the duration of the proof, we write our groups additively. Given a finite set of "non-laws" $\sum_{j=1}^{m} a_{ij}x_j \neq 0$, $1 \leq i \leq k$, of $\mathcal{A}$ with $m > 1$.

Set $p_i(x) = \sum_{j=1}^{m} a_{ij}x_j^{j-1}$ for $1 \leq i \leq k$. Let $M = \max_{i,j} |a_{ij}|$. Now

since every nonzero integral root of $p_i(x)$ must divide a_{ij_o} where $j_o = \min \{ j \in \mathbb{N} \mid a_{ij} \neq 0 \}$, we have $p_i(1+M) \neq 0$ for $1 \leq i \leq k$. It follows putting $a_1 = 1$, which is a free generator for $\mathbb{Z} = F_1(\mathcal{A})$, and $b_j = (1+M)^{j-1}$ for $2 \leq j \leq m$, that

$$a_{i1}a_1 + \sum_{j=2}^{m} a_{ij}b_j \neq 0 \text{ for } 1 \leq i \leq k$$

simultaneously in $\mathbb{Z}$. ■

Thus (B+3N)-discrimination and strong discrimination coincide for $\mathcal{A}$ since clearly

$$1 = \min \{ n \in \mathbb{N} \mid F_n(\mathcal{A}) \text{ (B+3N)-discriminates } \mathcal{A} \}.$$

Theorem 3 (N. Gupta and F. Levin): Let $c \in \mathbb{N}$. Let $r = \max\{2, c-1\}$. If $\mathcal{U}$ is any variety of groups whatsoever, then $F_r(\mathcal{U}\mathcal{N}_c)$ strongly discriminates $\mathcal{U}\mathcal{N}_c$.

Proof: Let $n \geq r = \max\{2,c-1\}$. Let $w_i \in F_{n+1}(\mathcal{U}\mathcal{N}_c) - \{1\}$, $1 \leq i \leq k$, be finitely many nontrivial elements of $F_{n+1}(\mathcal{U}\mathcal{N}_c)$. Then $\theta_m^* : F_{n+1}(\mathcal{U}\mathcal{N}_c) \longrightarrow F_n(\mathcal{U}\mathcal{N}_c)$, given in [13, Theorem 2], is a retraction which simultaneously does not kill off any w_i, $1 \leq i \leq k$. The proof is completed by taking the composition of the retractions found at each step

$$F_s(\mathcal{UN}_c) \longrightarrow F_{s-1}(\mathcal{UN}_c) \longrightarrow \cdots \longrightarrow F_r(\mathcal{UN}_c). \quad \blacksquare$$

Corollary 3.1 (N. Gupta and F. Levin): Let $c \in \mathbb{N}$ and $r = \max\{2, c-1\}$. Then $F_r(\mathcal{N}_c)$ strongly discriminates $\mathcal{N}_c$.

Proof: Put $\mathcal{U} = \mathcal{E}$ in Theorem 3. ∎

Corollary 3.2 (G. Baumslag): If $\mathcal{U}$ is any variety of groups, then $F_2(\mathcal{UA})$ strongly discriminates $\mathcal{UA}$.

Proof: Put $c = 1$ in Theorem 3. ∎

An independent argument can be given which is essentially the proof of [5, Theorem 3].

Corollary 3.3: If $k \in \mathbb{N}$ and $\mathcal{A}^k$ is the variety of all groups solvable of solvability length at most k, then $F_2(\mathcal{A}^k)$ strongly discriminates $\mathcal{A}^k$.

Proof: Put $\mathcal{U} = \mathcal{A}^{k-1}$ in Corollary 3.2 (where $\mathcal{A}^0 = \mathcal{E}$). ∎

Corollary 3.4 (N. Gupta and F. Levin): $F_2(\mathcal{A}^2)$ strongly discriminates $\mathcal{A}^2$.

Proof: Put $k = 2$ in Corollary 3.3. ∎

An independent argument may be given using [12, Lemma 2.3].

Corollary 3.5 (Sacerdote): $F_2(\mathcal{O})$ strongly discriminates $\mathcal{O}$.

Proof: Put $\mathcal{U} = \mathcal{O}$ in Corollary 3.2. ∎

Of course, this also follows from the fact that $F_r(\mathcal{O}) \le_{\forall} F_s(\mathcal{O})$ whenever $2 \le r \le s \le \omega$. If either $\mathcal{U}$ is not the trivial variety $\mathcal{E}$ or $c > 1$, then $\mathcal{UN}_c$ is not Abelian and

$$\min\{\, n \in \mathbb{N} \mid F_n(\mathcal{UN}_c) \text{ (B+3N)-discriminates } \mathcal{UN}_c \,\} \ge 2.$$

We have already seen that the concepts of (B+3N)-discrimination and strong discrimination coincide for the variety $\mathcal{EN}_1 = \mathcal{N}_1 = \mathcal{A}$ of Abelian groups. Clearly, $\min\{n \in \mathbb{N} \mid F_n(\mathcal{UA}) \text{ (B+3N)-discriminates } \mathcal{UA}\} = 2$ if $\mathcal{U} \ne \mathcal{E}$. Thus the concepts of (B+3N)-discrimination and strong discrimination coincide for every product variety $\mathcal{UA}$ whose

right hand factor is the variety $\mathcal{A}$ of all Abelian groups.

Definition 4: A group G is *discriminating* if it (B+3N)-discriminates the variety it generates.

It follows from Theorem 17.9 of [19] that torsion free nilpotent groups are discriminating. Thus, if $\mathcal{N}$ is a nilpotent variety whose free groups do not contain torsion, then

min{ $n \in \mathbb{N}$ | $F_n(\mathcal{N})$ (B+3N)-discriminates $\mathcal{N}$ } is

min{ $n \in \mathbb{N}$ | $F_n(\mathcal{N})$ generates $\mathcal{N}$ }. For the varieties $\mathcal{N}_c \wedge \mathcal{A}^2$ of nil-c and metabelian groups this number is 2 (except in the trivial case $\mathcal{N}_1 \wedge \mathcal{A}^2 = \mathcal{A}$ when it is 1). The above is the content of Theorem 36.34 of [19]. For varieties $\mathcal{N}_c$ of nil-c groups that number is c-1 whenever $c \geq 3$. Independent proofs of the above assertion may be found in [14] and [15]. Thus the concepts of (B+3N)-discrimination and strong discrimination coincide for the varieties $\mathcal{N}_c$ of nil-c groups. Now by 25.12 of [19] if $\mathcal{U} \neq \mathcal{O}$ and $\mathcal{UV}$ is generated by its n-generator groups, then so is $\mathcal{V}$. It follows that for $\mathcal{U} \neq \mathcal{O}$, except in the trivial case $\mathcal{U} = \mathcal{E}$ and $c = 1$ simultaneously, r = max {2, c-1}

= min{ $n \in \mathbb{N}$ | $F_n(\mathcal{UN}_c)$ generates $\mathcal{UN}_c$ }

= min{ $n \in \mathbb{N}$ |$F_n(\mathcal{UN}_c)$(B+3N)-discriminates $\mathcal{UN}_c$}.

Observe that the proviso that $\mathcal{U} \neq \mathcal{O}$ was incorrectly omitted in [10]. Thus, the concepts of (B+3N)-discrimination and strong discrimination coincide for any product variety $\mathcal{UN}_c$.

Theorem 4: $F_2(\mathcal{N}_c \wedge \mathcal{A}^2)$ strongly discriminates $\mathcal{N}_c \wedge \mathcal{A}^2$.

Outline of Proof: $F_2(\mathcal{N}_1 \wedge \mathcal{A}^2) = F_2(\mathcal{A}) = \mathbb{Z}^2$. Not only does $\mathbb{Z}^2$ strongly discriminate $\mathcal{A}$ but $\mathbb{Z}$ already does.

We use induction on c. Suppose the result is true for c-1. Let $w_i(a_1,\ldots,a_m) \neq 1$ with $m \geq 2$ and $1 \leq i \leq k$ be a finite set of

"non-laws" of $\mathcal{N}_c \wedge \mathcal{A}^2$ expressed as inequations in $G = F_m(\mathcal{N}_c \wedge \mathcal{A}^2)$ which is $\mathcal{N}_c \wedge \mathcal{A}^2$ freely generated by $a_1 < a_2 < \ldots < a_m$. Let $F_2(\mathcal{N}_c \wedge \mathcal{A}^2) = gp(a_1,a_2)$. Then w_i is uniquely expressible as a product of powers of left-normed basic commutators where the factors are arranged according to the ordering of basic commutators. This last assertion is the content of Theorem 36.32 of [19]. Let Γ be the set of left-normed basic commutators in $a_1,\ldots,a_m$ and $\Gamma_o \subseteq \Gamma$ be the set of left-normed basic commutators in a_1,a_2. Then we have

$$w_i = \prod_{\gamma\in\Gamma} \gamma(a_1,\ldots,a_m)^{e_{i\gamma}} \text{ (appropriate order understood).}$$

Let ψ be the retraction

$$\psi : F_m(\mathcal{N}_c \wedge \mathcal{A}^2) \longrightarrow F_2(\mathcal{N}_c \wedge \mathcal{A}^2)$$

given by

$$\begin{cases} a_1 \longmapsto a_1 \\ a_2 \longmapsto a_2 \\ a_\nu \longmapsto \prod_{\gamma\in\Gamma_o} \gamma(a_1,a_2)^{x_{\gamma\nu}} & \text{for } 3 \le \nu \le m \\ & \text{(appropriate order understood)} \end{cases}$$

Then

$$\psi(w_i) = \prod_{\gamma\in\Gamma_o} \gamma(a_1,a_2)^{\varepsilon_{i\gamma}}$$

for $1 \le i \le k$ (again in the appropriate order).

Form the polynomial $f = \prod_{i=1}^{k} \left(\sum_{\gamma\in\Gamma_o} \varepsilon_{i\gamma}^2 \right)$ in the $x_{\gamma\nu}$. Then $f \neq 0$ whenever ψ serves our purposes and conversely. Hence each factor of f is not identically zero in this case and we hereby assume that k = 1 and drop the subscript i. Thus we must show that

$$w = \prod_{\gamma\in\Gamma} \gamma(a_1,\ldots,a_m)^{e_\gamma} \neq 1$$

implies that

$$\psi(w) = \prod_{\gamma\in\Gamma_o} \gamma(a_1,a_2)^{\varepsilon_\gamma} \neq 1$$

for appropriate choice of ψ.

Suppose that $w \not\equiv 1(\bmod \gamma_c(G))$. Then $w(x_1,\ldots,x_m) \neq 1$ is a "non-law" in $\mathcal{N}_{c-1} \wedge \mathcal{A}^2$ since $G/\gamma_c(G) = F_m(\mathcal{N}_{c-1} \wedge \mathcal{A}^2)$. By inductive hypothesis, $F_2(\mathcal{N}_{c-1} \wedge \mathcal{A}^2)$ strongly discriminates $\mathcal{N}_{c-1} \wedge \mathcal{A}^2$. Hence we can find ψ such that $\psi(w) \neq 1$ in this case. We may therefore assume $w \in \gamma_c(G)$. We now use induction on m. If $m = 2$, then $w(a_1,a_2)$ is an element of $gp(a_1,a_2) - \{1\}$ and we are finished. The inductive hypothesis is: For every $w_0 \in gp(a_1,\ldots a_{m-1}) - \{1\}$, there is a retraction $\psi : gp(a_1,\ldots a_{m-1}) \longrightarrow gp(a_1,a_2)$ such that $\psi(w_0) \neq 1$.

Let D be the set of all left-normed basic commutators of weight c in G which contain a_3 and E be the set of all left-normed basic commutators of weight c in G which do not contain a_3. Then $w(a_1,\ldots,a_m) = u(a_1,\ldots,a_m)v(a_1,\ldots,a_m)$ where $u = \prod_{\beta\in D} \beta^{e_\beta}$ and $v = \prod_{\beta\in E} \beta^{e_\beta}$. Here the order of the factors is not important because $\gamma_c(G)$ lies in the center of G. Let π be the transposition (3 m) and $\alpha : G \longrightarrow G$ the automorphism $a_\nu \longmapsto a_{\pi(\nu)}$ induced by π. If $v \neq 1$, then $\alpha(v) \neq 1$. Moreover, $\alpha(v)$ is an element of

$gp(a_1,\ldots,a_{m-1})$. Thus there is a retraction

$\phi : gp(a_1,\ldots,a_{m-1}) \longrightarrow gp(a_1,a_2)$ such that $\phi\alpha(v) \neq 1$. Define

$\psi : G \longrightarrow gp(a_1,a_2)$ by

$$\begin{cases} \psi(a_\nu) = \phi\alpha(a_\nu) & \text{if } \nu \neq 3 \\ \psi(a_3) = 1 \end{cases}$$

Then $\psi(w) = \psi(uv) = \psi(u)\psi(v) = 1\cdot\phi\alpha(v) = \phi\alpha(v) \neq 1$.

Thus we may assume that w is a product of left-normed basic commutators of weight c in G all of which involve a_3. In particular, if $w = \prod_\beta \beta^{e_\beta}$ with $\beta = [a_{\beta_1},\ldots,a_{\beta_c}]$, then we may assume that each $\beta_2 \leq 3$. We shall complete the double induction

and prove the theorem by inferring for each $w = \prod_{\beta} \beta^{e_\beta} \in \gamma_c(G)-\{1\}$ such that each $\beta_2 \le 3$ the existence of a retraction

$$\psi : G \longrightarrow F_{m-1}(\mathcal{N}_c \wedge \mathcal{A}^2)$$

with the property that $\psi(w) \neq 1$. The retractions deduced to exist shall be of the form

$$\begin{cases} a_\nu \longmapsto a_\nu & \text{for } \nu = 1,2,\ldots,m-1 \\ a_m \longmapsto a_{m-2}^x\, a_{m-1}^y \end{cases}$$

It suffices to consider four cases, viz.,

Case I: $m = 3$

Case II: $m = 4$

Case III: $m = 5$

Case IV: $m \ge 6$.

In each case, the argument proceeds as follows. If $w = \prod_{\beta} \beta^{e_\beta}$ with each $\beta_2 \le 3$, then $\psi(w) = \prod_{\gamma} \gamma^{\varepsilon_\gamma}$ where the γ vary over the left-normed basic commutators of weight c in the m-1 $\mathcal{N}_c \wedge \mathcal{A}^2$ free generators $a_1,\ldots,a_{m-1}$ of $gp(a_1,\ldots,a_{m-1})$ and the exponents ε_γ are polynomials in the x and y with coefficients which themselves are linear forms over $\mathbb{Z}$ in the exponents e_β of the basic commutators β in $a_1,\ldots,a_{m-1},a_m$. We isolate (in some cases making simplifying assumptions following prior deductions) nonzero integral multiples of the e_β as coefficients of the terms $x^p y^q$ in some ε_γ. It then follows that we can assume $e_\beta = 0$; for otherwise, ε_γ is not the zero polynomial and we can find x and y such that $\varepsilon_\gamma \neq 0$ and so $\psi(w) \neq 1$ since $\psi(w)$ contains a factor $\gamma^{\varepsilon_\gamma} \neq 1$. Finally upon deducing that every $e_\beta = 0$, we conclude that the only element which is killed off by every retraction is the identity, 1, of G. That completes the double induction in each case and so proves

the theorem. ∎

It follows that the concepts of (B+3N)-discrimination and strong discrimination coincide for the varieties $\mathcal{N}_c \wedge \mathcal{A}^2$. We have to date not discovered a variety of groups in which these two concepts do not coincide. Perhaps the Burnside varieties are candidates for such pathology? Let n be odd with $n \geq 665$. Since $F_2(\mathcal{B}_n) \leq F_s(\mathcal{B}_n)$for all s with $2 \leq s \leq \omega$, we must have that every universal sentence true in $F_s(\mathcal{B}_n)$ is true in $F_2(\mathcal{B}_n)$ for all such s. Adian has shown that there is an embedding $F_3(\mathcal{B}_n) \longrightarrow F_2(\mathcal{B}_n)$ and Širvanjan [25] has shown that there is an embedding $F_\omega(\mathcal{B}) \longrightarrow F_3(\mathcal{B}_n)$ from which it follows that $F_\omega(\mathcal{B}_n)$ satisfies precisely the same universal sentences as $F_2(\mathcal{B}_n)$ for all s with $2 \leq s \leq \omega$. In other words, $F_2(\mathcal{B}_n) \ \forall \ F_s(\mathcal{B}_n)$ for all $s \geq 2$ or $F_2(\mathcal{B}_n)$ (B+3N)-discriminates $\mathcal{B}_n$ whenever $n \geq 665$ is odd.

4. Reflections on the Model Class of the Theory of the non-Abelian Absolutely Free Groups

In this section and the following one, we shall use the term "free group" in its usual sense of absolutely free group - i.e., free in the variety of all groups. Let $\sum$ be the set of all sentences σ of L_o such that σ is true in $F_r(\mathcal{O})$ for all r with $2 \leq r \leq \omega$. $\sum$ is the *theory of the non-Abelian free groups*. Tarski has conjectured that $\sum$ is decidable. Suppose we expand L_o to L_o^+ by adjoining two new constant symbols a and b. Let τ be the sentence $[a,b] = 1$ of L_o^+ and for each $(m,n) \in \mathbb{Z}^2$ let $\tau(m,n)$ be the sentence

$$\sim \exists x \left[(a = x^m) \wedge (b = x^n) \right]$$

of L_o^+. Let $T = \{\tau\} \cup \{\tau(m,n) \mid (m,n) \in \mathbb{Z}^2\}$. Since every finite

subset of $\sum \cup T$ has a model, we see by the Compactness Theorem and the Lowenheim-Skolem Theorem (see [7]) that $\sum \cup T$ must have a countable model G_o. G_o cannot be a free group since, for example, the centralizer of the nontrivial element (the interpretation in G_o of) a is not cyclic. Thus, there are non-free, even countable non-free, models of $\sum$.

A group G is *r-free* ($r \in \mathbb{N}$) if every r-generator subgroup[1] of G is free. We shall presently show that there are finitely generated non-Abelian 2-free groups which are not models of $\sum$ and that for all integral $r \geq 3$ there are non-Abelian r-free groups which are not models of $\sum$.

Question: Does there exist an integer $r \geq 3$ such that every finitely generated non-Abelian r-free group is a model of $\sum$? In particular, is this condition satisfied by $r = 3$?

Let $G = F_2(\ D(\{2\})\ )$ be a free $D(\{2\})$-algebra of rank 2 freely generated by $\{a_1,a_2\}$. B. Baumslag [2] has shown that the free algebras $F_r(D(P))$ are 2-free. Since $F_2(\ D(\{2\})\ )$ is embedded as a subgroup of $F_2(D(P))$, we conclude that G is also 2-free because r-freeness is a hereditary property. Let H be the subgroup of G generated (as a group) by a_1,a_2 and $[a_1,a_2]^{\frac{1}{2}}$. Then H is 2-free but H is not a model of $\sum$ since

$$\exists\, x\ \exists y\ \exists z \left[\ ([x,y] \neq 1) \wedge ([x,y] = z^2)\ \right]$$

is true in H, but false in every non-Abelian free group by a result of Schützenberger [24]. Put for each $n < \omega$ K_n to be the subgroup of G generated (as a group) by $\{b^{(n)},a_2\}$ where $b^{(n)}$ is the 2^n-th root

[1] A group is r-generator if it can be generated by r or fewer distinct elements.

of a_1. Put $K = \bigcup_{n<\omega} K_n$. Then every finitely generated subgroup of K is free so that K is r-free for every $r \geq 3$, but K is not a model of $\sum$ since

$$\exists\, x \left[(x \neq 1) \wedge \forall\, y(([x,y] = 1) \Rightarrow \exists\, z\, (y = z^2)) \right]$$

is true in K but false in every non-Abelian free group. This is so because the centralizers of nontrivial elements of free groups are infinite cyclic.

Definition 4: A group is *Sacerdotian* if it decomposes as a nontrivial free product and is not isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$. The group G is *positive* if it has a Sacerdotian homomorphic image.

Lemma 4· The positive groups satisfy precisely the same positive and negative sentences of L_o.

Proof: Let G be positive. Then G has a Sacerdotian homomorphic image H. In particular, G is non-Abelian. Hence G may be presented as the homomorphic image of a non-Abelian free group F. Now every positive sentence true in G must be true in H by Lyndon's Theorem. Similarly, every positive sentence true in F must be true in G. But F and H satisfy precisely the same positive sentences by Sacerdote's results. It follows that F and G satisfy precisely the same positive and negative sentences of L_o. ■

Now let A and B be disjoint sets having cardinals r and s, respectively. Suppose further that the $\min\{r,s\} \geq 2$. Let F be a free group of rank r with free basis A and let $\bar{F}$ be a free group of rank s with free basis B. Suppose $u \in F$ is neither a primitive element nor a proper power and $v \in \bar{F}$ is neither a primitive element nor a proper power. Then the centralizer of u in F is $U = \langle u^n \mid n \in \mathbb{Z} \rangle$ and the centtralizer of v in $\bar{F}$ is $V = \langle v^n \mid n \in \mathbb{Z} \rangle$. The map $\beta : U \quad V$, $u^n \longmapsto v^n$ for all $n \in \mathbb{Z}$ is an isomorphism. Following G. Baumslag

[4], we use the notation

$$(F * \bar{F};\ u = v)$$

for the generalized free product of F with $\bar{F}$ amalgamating U and V via β. Rosenberger [21] has shown that such $(F * \bar{F};\ u = v)$ are 3-free. Now suppose that $r = s$ and $B = \bar{A}$ is in bijective correspondence with A via $a_\alpha \longmapsto \bar{a}_\alpha$ for all $\alpha < r$. Let $\psi : F \quad \bar{F}$ be the isomorphism induced by this bijection and suppose further that $v = \bar{u} = \psi(u)$. We shall call a group $K = (F * \bar{F};\ u = \bar{u})$ a *Baumslag group* after G. Baumslag who studied them in [4].

Observe that the map

$$\begin{cases} a_\alpha \longmapsto a_\alpha \\ \bar{a}_\alpha \longmapsto a_\alpha \end{cases}$$

on the generators $A \cup \bar{A}$ of K preserves the relation $u = \bar{u}$ so extends to a retraction $K \longrightarrow F$. It follows that K is positive so K satisfies precisely the same positive and negative sentences of L_o as the non-Abelian free groups.

Even more is true. To continue we first state that in this section, we shall refer liberally to G. Baumslag [4] without further attribution. Suppose $r \geq 2$ is finite. We have the embeddings

$$F \longrightarrow K \longrightarrow H \longrightarrow F^{\mathbb{N}} = J.$$

(We can take each ${}_iF = F$ and each isomorphism ${}_iF \cong F$ to be the identity map.) Here $H = (F * G;\ u = v)$ where G is free Abelian of countably infinite rank with basis $X \cup \{v\}$. Moreover we identify F with its image in $K = gp(F,\ x_o^{-1}Fx_o)$ where $x_o \in X$. Letting for each $i \in \mathbb{N}$, θ_i be the natural projection $\theta_i : J \longrightarrow F$ onto the i-th coordinate, we see that we have a homomorphism $\varepsilon_i : K \longrightarrow F$ making the triangle

$$\begin{tikzcd} K \arrow[r] \arrow[dr, "\varepsilon_i"'] & J \arrow[d, "\theta_i"] \\ & F \end{tikzcd}$$

commutative.

If $f \in F \leq K$, then the image $\hat{f}$ of f in J is the constant sequence $(f,f,\ldots,f,\ldots)$. Hence, $\varepsilon_i(f) = \theta_i(\hat{f}) = f$.

Theorem 5: $F \leq_V K$.

Proof: Define for each $k \in K$ the *support* of k by $\mathrm{Supp}(k) = \{i \in \mathbb{N} \mid \varepsilon_i(k) \neq 1\}$. We claim that for each $k \neq 1$ in K, $\mathrm{Supp}(k)$ is a cofinite subset of $\mathbb{N}$ (i.e., $\mathbb{N} - \mathrm{Supp}(k)$ is finite if $k \neq 1$.). Granting the claim momentarily, it follows that

$$\left\{ \mathrm{Supp}(k) \mid k \in K - \{1\} \right\}$$

satisfies the finite intersection property. Thus given finitely many $k_\nu \neq 1$, $1 \leq \nu \leq n$, nontrivial elements of K and choosing $i_o \in \mathrm{Supp}(k_1) \cap \ldots \cap \mathrm{Supp}(k_n)$, we have the retraction $\varepsilon_{i_o} : K \quad F$ such that $\varepsilon_{i_o}(k_\nu) \neq 1$ for $\nu = 1,2,\ldots,n$. It then follows from Lemma 2 that $F \leq_V K$. Thus it remains to verify the claim.

Suppose $k \neq 1$ lies in K. Then using properties of generalized free products, we see that the image of k in J has the form

$$\hat{f}_1 g_1 \hat{f}_2 g_2 \ldots \hat{f}_n g_n u^p$$

where $n \geq 0$ and we may insist that

1. If $n = 0$, then $p \neq 0$.
2. $\hat{f}_2,\ldots,\hat{f}_n \notin gp(u)$ and $g_1,\ldots,g_{n-1} \notin gp(u)$.
3. Either $\hat{f}_1 = 1$ or $\hat{f}_1 \notin gp(u)$ and either $g_n = 1$ or $g_n \notin gp(u)$.
4. If $n = 1$, then not both $\hat{f}_1 = 1$ and $g_n = 1$ can hold.

Conjugating k (if necessary) by an appropriate element of H, we may apply G. Baumslag's Proposition 1 and argue similarly to him that $\varepsilon_i(k)$ could not be trivial for infinitely many values of i. Hence

Supp(k) is cofinite whenever $k \neq 1$ as claimed. ■

Remark: The same argument shows that if $H = (F * G;\ u = v)$ is as in [4], then $F \leq_V H$; moreover, each $\theta_i|_H$ is a retraction so H is positive.

5. The Equation $[y_1, y_2] = x_1^2 x_2^2$ in Free Groups

Consider the group G with presentation

$$\langle a_1, a_2, b_1, b_2;\ a_1^{-1} a_2^{-1} a_1 a_2 = b_1^2 b_2^2 \rangle.$$

Putting

$$\begin{cases} s_1 = a_1^{-1} \\ s_2 = b_2^{-1} \\ t_1 = a_2^{-1} \\ t_2 = b_2^{-1} b_1^{-1} \end{cases}$$

We see that the "st-coordinates" are obtained by means of a Neislen transformation whose inverse is given by

$$\begin{cases} a_1 = s_1^{-1} \\ a_2 = t_1^{-1} \\ b_1 = t_2^{-1} s_2 \\ b_2 = s_2^{-1}. \end{cases}$$

Furthermore, the relation $a_1^{-1} a_2^{-1} a_1 a_2 = b_1^2 b_2^2$ in the "ab-coordinates" is equivalent to $s_1 t_1 s_1^{-1} t_1^{-1} s_2 t_2 s_2^{-1} t_2 = 1$ in the "st-coordinates". Thus G is isomorphic to

$$\langle s_1, t_1, s_2 t_2;\ s_1 t_1 s_1^{-1} t_1^{-1} s_2 t_2 s_2^{-1} t_2 = 1 \rangle.$$

But by a result stated in Comerford and Lee [8],

$\langle s_1, t_1, s_2 t_2;\ s_1 t_1 s_1^{-1} t_1^{-1} s_2 t_2 s_2^{-1} t_2 = 1 \rangle$ is isomorphic to

$\langle x_1,x_2,x_3,x_4;\ x_1^2x_2^2x_3^2x_4^2 = 1\rangle$ which is obviously isomorphic to the Baumslag group $\langle a_1,a_2,\bar{a}_1,\bar{a}_2;\ a_1^2a_2^2 = \bar{a}_1^2\bar{a}_2^2\rangle$. Thus not only is G 3-free, which follows from the original presentation by the result of Rosenberger previously alluded to, but G is of the form $(F * \bar{F};\ u = \bar{u})$ where u is not a proper power in $F = \langle a_1,a_2\rangle$. We wish to ponder whether or not G is a model of $\sum$. To this end, we study the equation $[y_1,y_2] = x_1^2x_2^2$ in F. It is known that $[a_1,a_2] = x_1^2x_2^2$ has no solution in F. (See [9] or [17].)

Definition 5: A *form* ϕ is an ordered pair $\phi = (\phi_1,\phi_2)$ of elements of the group $\langle x,y,z;\ [y,z] = 1\rangle$. An ordered pair (f_1,f_2) of elements of F is *of the form* ϕ provided there is at least one homomorphism $h : gp(\phi_1,\phi_2) \longrightarrow F$ such that $h(\phi_i) = f_i$ $(i=1,2)$.

Now suppose (c,d) is an ordered pair of non-commuting elements of F. Suppose further that $[c,d] = x_1^2x_2^2$ has a solution in F. At first the authors did not discover any such pair (c,d) not of at least one of the forms (zx^2,y) or (y,zx^2) - giving solutions $[c,d] = e^2f^2$ where $e = x^{-1}$ and $f = y^{-1}xy$ in the first instance and $e = y^{-1}x^{-1}y$ and $f = x$ in the second. Note that with these choices for e and f, we have in either event that f is a conjugate of e^{-1} whence ef is a commutator. The authors have discovered a solution of neither of the above forms and have also found a solution $[c,d] = e^2f^2$ in which ef is not a commutator.

For $u \neq 1$ a nontrivial element of the commutator subgroup F' of F, let its *genus* be defined by

$$\text{genus}(u) = \min\left\{ n \in \mathbb{N} \mid u = \prod_{i=1}^{n} [x_i,y_i] \right\}.$$

Now let Φ be a finite set of forms and let $n \in \mathbb{N}$. Consider the Π_2-sentences of L_o

$$\forall x_1 \forall x_2 \forall y_1 \forall y_2 \exists z_1 \ldots \exists z_n \exists w_1 \ldots \exists w_n \left[([y_1,y_2] \neq 1 \wedge [y_1,y_2] = x_1^2 x_2^2) \Rightarrow \right.$$

$$\left. (x_1 x_2 = \prod_{i=1}^{n} [z_i, w_i]) \right] \qquad (*)_n$$

and

$$\forall x_1 \forall x_2 \forall y_1 \forall y_2 \exists x \exists y \exists z \left[([y,z]=1 \wedge [y_1,y_2] \neq 1 \wedge [y_1,y_2] = x_1^2 x_2^2) \Rightarrow \right.$$

$$\left. \bigvee_{\phi \in \Phi} (y_1 = \phi_1 \wedge y_2 = \phi_2) \right] \qquad (**)_\Phi$$

We shall show that $(*)_n$ is false in G for every n. The assertion that, for some n, $(*)_n$ is true in F may be paraphrased as : There is a bound on genus(ef) for all solutions $[c,d] = e^2 f^2 \neq 1$ in F. Thus if there is such a bound, then G would be a Baumslag group which is not a model of $\sum$ because $(*)_n$ would be true in F but not true in G as we shall presently show. Let G' be the commutator subgroup of G and $\bar{G}$ be the Abelianization of G. Let $\rho : G \longrightarrow \bar{G}$ be the natural projection $g \longmapsto \bar{g}$. Then, as an Abelian group, $\bar{G}$ has the presentation

$$\langle \bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_1 \bar{b}_2;\ (\bar{b}_1 \bar{b}_2)^2 = [u,v] = 1 \text{ for all } u,v \rangle.$$

So $\bar{G} \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2$ and $\bar{b}_1 \bar{b}_2$ is the unique element of order 2 in $\bar{G}$. In particular, $b_1 b_2 \notin G'$ so $(*)_n$ is false in G for all n as claimed.

6. Open Questions

1. (Tarski) If $2 \leq r \leq s$, is $F_r(\mathcal{O}) \equiv F_s(\mathcal{O})$?
2. (Tarski) If $2 \leq r \leq s$, is $F_r(\mathcal{O}) \propto F_s(\mathcal{O})$?
3. If P_o is a set of primes and $2 \leq r \leq s$, is $F_r(D(P_o)) \equiv F_s(D(P_o))$?
4. If P_o is a set of primes and $2 \leq r \leq s$, is $F_r(D(P_o)) \propto F_s(D(P_o))$?
5. If $n \geq 665$ and is odd, does $F_2(\mathcal{B})$ strongly discriminate $\mathcal{B}_n$?

6. If $n \geq 665$, is odd, and $2 \leq r \leq s$, is $F_r(\mathcal{B}) \equiv F_s(\mathcal{B})$?

7. If $n \geq 665$, is odd, and $2 \leq r \leq s$, is $F_r(\mathcal{B}) \propto F_s(\mathcal{B})$?

8. Do (B+3N)-discrimination and strong discrimination coincide for the free algebras of every nontrivial variety $\mathcal{V}$?

9. (P. M. Neumann) Same as question 8 for the special case of nilpotent varieties $\mathcal{N}$ of groups whose free groups do not contain torsion.

10. (Tarski) Is the theory $\sum$ of the non-Abelian free groups decidable?

11. Does there exist an integer $r \geq 3$ such that every finitely generated non-Abelian r-free group is a model of $\sum$? In particular, does $r = 3$ satisfy this condition?

12. If $K = (F * \bar{F};\ u = \bar{u})$ (u not a proper power in the non-Abelian free group F) is a Baumslag group, is K a model of $\sum$?

13. Is every finitely generated model of $\sum$ free?

14. Is there a bound on the genus(ef) for all solutions to $[c,d] = e^2f^2 \neq 1$ (valid for all non-commuting pairs (c,d) and all solutions $x_1 = e$ and $x_2 = f$ to $[c,d] = x_1^2x_2^2$) in a free group F?

15. (L. Comerford and C.C. Edmunds) Let F be a non-Abelian free group with commutator subgroup F'. Let $U \in F' - \{1\}$. Is the genus(U^n) a non-decreasing function of n?

16. (L. Comerfordand C.C. Edmunds) Same hypotheses as in 15. Is there a bound on genus(U^m), $m \leq n$, in terms of genus(U^n) that is independent of U?

17. Let F bc a non-Abelian free group and let (c,d) be an ordered pair of non-commuting elements of F such that the equation $[c,d] = x_1^2x_2^2$ has a solution in F. Must there exist a finite

set Φ of "forms" such that every pair is "of the form ϕ" for at least one $\phi \in \Phi$? (Note that here we are using the terminology of Definition 5.)

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